

# A SCHOENFLIES EXTENSION THEOREM FOR A CLASS OF LOCALLY BI-LIPSCHITZ HOMEOMORPHISMS

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**ABSTRACT.** In this paper we prove a new version of the Schoenflies extension theorem for collared domains  $\Omega$  and  $\Omega'$  in  $\mathbb{R}^n$ : for  $p \in [1, n)$ , locally bi-Lipschitz homeomorphisms from  $\Omega$  to  $\Omega'$  with locally  $p$ -integrable, second-order weak derivatives admit homeomorphic extensions of the same regularity.

Moreover, the theorem is essentially sharp. The existence of exotic 7-spheres shows that such extension theorems cannot hold, for  $p > n = 7$ .

## 1. INTRODUCTION

**1.1. Embeddings of Collars.** In point-set topology, the Schoenflies Theorem [Wil79, Thm III.5.9] is a stronger form of the well-known Jordan Curve Theorem: it states that *every simple closed curve separates the sphere  $\mathbb{S}^2$  into two domains, each of which is homeomorphic to  $\mathbb{B}^2$ , the open unit disc.* The same statement does not hold in higher dimensions, since the Alexander horned sphere [Ale24] provides a counter-example in  $\mathbb{R}^3$ . Despite this, Brown [Bro60] proved that for each  $n \in \mathbb{N}$ , every embedding of  $\mathbb{S}^{n-1} \times (-\epsilon, \epsilon)$  into  $\mathbb{R}^n$  extends to an embedding of  $\mathbb{B}^n$  into  $\mathbb{R}^n$ .

Similar extension problems arise by varying the regularity of the embeddings. To this end, we prove a Schoenflies-type theorem for a new class of homeomorphisms. Their regularity is given in terms of Sobolev spaces and Lipschitz continuity.

To begin, recall that a homeomorphism  $f : \Omega \rightarrow \Omega'$  is *locally bi-Lipschitz* if for each  $z \in \Omega$ , there is a neighborhood  $O$  of  $z$  and  $L \geq 1$  so that the inequality

$$(1.1) \quad L^{-1} |x - y| \leq |f(x) - f(y)| \leq L |x - y|$$

holds for all  $x, y \in O$ . Recall also that for  $p \geq 1$  and  $k \in \mathbb{N}$ , the Sobolev space  $W_{\text{loc}}^{k,p}(\Omega; \Omega')$  consists of maps  $f : \Omega \rightarrow \Omega'$ , where each component  $f_i$  lies in  $L_{\text{loc}}^p(\Omega)$  and has weak derivatives of orders up to  $k$  in  $L_{\text{loc}}^p(\Omega)$ .

**Definition 1.1.** Let  $f : \Omega \rightarrow \Omega'$  be a locally bi-Lipschitz homeomorphism. For  $p \in [1, \infty)$ , we say that  $f$  is of *class  $LW_2^p$*  if  $f \in W_{\text{loc}}^{2,p}(\Omega; \Omega')$  and  $f^{-1} \in W_{\text{loc}}^{2,p}(\Omega'; \Omega)$ . If  $K$  and  $K'$  are closed sets, a homeomorphism  $f : K \rightarrow K'$  is of class  $LW_2^p$  if the restriction of  $f$  to the interior of  $K$  is of class  $LW_2^p$ .

Instead of product sets of the form  $\mathbb{S}^{n-1} \times (-\epsilon, \epsilon)$ , we will consider domains in  $\mathbb{R}^n$  of a similar topological type.

**Definition 1.2.** A bounded domain  $D$  in  $\mathbb{R}_*^n$  is *Jordan* if its boundary  $\partial D$  is homeomorphic to  $\mathbb{S}^{n-1}$ . A *collared domain* (or *collar*) is a domain in  $\mathbb{R}^n$  of the form  $D_2 \setminus \bar{D}_1$ , where  $D_1$  and  $D_2$  are Jordan domains with  $\bar{D}_1 \subset D_2$ .

We now state the extension theorem for homeomorphisms of class  $LW_2^p$  between collared domains.

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**Theorem 1.3.** *Let  $D_1$  and  $D_2$  be Jordan domains in  $\mathbb{R}^n$  so that  $\bar{D}_1 \subset D_2$ , let  $B_1$  and  $B_2$  be balls so that  $\bar{B}_1 \subset B_2$ , and let  $p \in [1, n)$ .*

*If  $f : \bar{D}_2 \setminus D_1 \rightarrow \bar{B}_2 \setminus B_1$  is a homeomorphism of class  $LW_2^p$  so that  $f(\partial D_i) = \partial B_i$  holds, for  $i = 1, 2$ , then there exists a homeomorphism  $F : \bar{D}_2 \rightarrow \bar{B}_2$  of class  $LW_2^p$  and a neighborhood  $N$  of  $\partial D_2$  so that  $F|(N \cap \bar{D}_2) = f|(N \cap \bar{D}_2)$ .*

The proof is an adaptation of Gehring's argument [Geh67, Thm 2'] from the class of quasiconformal homeomorphisms to the class  $LW_2^p$ . For the locally bi-Lipschitz class, the extension theorem was known to Sullivan [Sul75] and later proved by Tukia and Väisälä [TV81, Thm 5.10]. For more about quasiconformal homeomorphisms, see [Väi71].

As in Gehring's case, Theorem 1.3 is not quantitative. His extension depends on the distortion (resp. Lipschitz constants) of  $g$  as well as the configurations of the collars  $D_2 \setminus \bar{D}_1$  and  $B_2 \setminus \bar{B}_1$ . In addition, our modification of his extension also depends explicitly on the parameters  $p$  and  $n$ .

**1.2. Motivations, Smoothness, and Sharpness.** The motivation for Theorem 1.3 comes from the study of Lipschitz manifolds. Specifically, Heinonen and Keith have recently shown that *if an  $n$ -dimensional Lipschitz manifold, for  $n \neq 4$ , admits an atlas with coordinate charts in the Sobolev class  $W_{loc}^{2,2}(\mathbb{R}^n; \mathbb{R}^n)$ , then it admits a smooth structure* [HK09].

On the other hand, there are 10-dimensional Lipschitz manifolds without smooth structures [Ker60]. This leads to the following question:

**Question 1.4.** For  $n \neq 4$ , does there exist  $p \in [1, 2)$  so that every  $n$ -dimensional Lipschitz manifold admits an atlas of charts in  $W_{loc}^{2,p}(\mathbb{R}^n; \mathbb{R}^n)$ ?

Sullivan has shown that *every  $n$ -dimensional topological manifold, for  $n \neq 4$ , admits a Lipschitz structure* [Sul75]. A key step in the proof is to show that bi-Lipschitz homeomorphisms satisfy a Schoenflies-type extension theorem. One may inquire whether this direction of proof would also lead to the desired Sobolev regularity. Theorem 1.3 would be a first step in this direction. For more about Lipschitz structures on manifolds, see [LV77].

It is worth noting that Theorem 1.3 is not generally true for  $p > n$ . Recall that for any domain  $\Omega$  in  $\mathbb{R}^n$ , Morrey's inequality [EG92, Thm 4.5.3.3] gives  $W^{2,p}(\Omega) \hookrightarrow C^{1,1-n/p}(\Omega)$ , so homeomorphisms of class  $LW_2^p$  are necessarily  $C^1$ -diffeomorphisms.

Indeed, every  $C^\infty$ -diffeomorphism  $\varphi : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$  admits a radial extension

$$\bar{\varphi}(x) := |x| \varphi\left(\frac{x}{|x|}\right)$$

that is, a  $C^\infty$ -diffeomorphism between round annuli. The validity of Theorem 1.3, for  $p > n$ , would therefore imply that every such  $\varphi$  extends to a  $C^1$ -diffeomorphism of  $\mathbb{B}^n$  onto itself. However, for  $n = 7$  this conclusion is impossible.

Recall that every such  $\varphi$  also determines a  $C^\infty$ -smooth,  $n$ -dimensional manifold  $M_\varphi^n$  that is homeomorphic to  $\mathbb{S}^n$  [Mil56, Construction (C)]. Indeed,  $M_\varphi^n$  is the quotient of two copies of  $\mathbb{R}^n$  under the relation  $x \sim \varphi^*(x)$  on  $\mathbb{R}^n \setminus \{0\}$ , where

$$(1.2) \quad \varphi^*(x) := \frac{1}{|x|} \varphi\left(\frac{x}{|x|}\right).$$

If  $\varphi$  is the identity map on  $\mathbb{S}^{n-1}$ , then  $\varphi^*$  is the inversion map  $x \mapsto |x|^{-2}x$ , and  $M_\varphi^n$  is precisely  $\mathbb{S}^n$ . By using invariants from differential topology, Milnor proved the following theorem about such manifolds [Mil56, Thm 3].

**Theorem 1.5** (Milnor, 1956). *There exist  $C^\infty$ -smooth manifolds of the form  $M_\varphi^7$  that are homeomorphic, but not  $C^\infty$ -diffeomorphic, to  $\mathbb{S}^7$ .*

Such manifolds are better known as *exotic spheres*. The next lemma is an analogue of [Hir94, Thm 8.2.1]; it relates exotic spheres to extension theorems.

**Lemma 1.6.** *Let  $\varphi : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$  be a  $C^\infty$ -diffeomorphism and let  $\bar{\varphi} : \mathbb{B}^n \setminus \{0\} \rightarrow \mathbb{B}^n \setminus \{0\}$  be its radial (diffeomorphic) extension. If there exists a  $C^1$ -diffeomorphism  $\Phi : \mathbb{B}^n \rightarrow \mathbb{B}^n$  that agrees with  $\bar{\varphi}$  on a neighborhood of  $\mathbb{S}^{n-1}$  in  $\mathbb{B}^n$ , then  $M_\varphi^n$  is  $C^1$ -diffeomorphic to  $\mathbb{S}^n$ .*

*Proof of Lemma 1.6.* Let  $\varphi^*$  be the diffeomorphism defined in Equation (1.2). By construction, there is an atlas of charts  $\{M_i\}_{i=1}^2$  for  $M_\varphi^n$  with homeomorphisms  $\psi_i : M_i \rightarrow \mathbb{R}^n$  that satisfy  $\psi_1 \circ \psi_2^{-1} = \varphi^*$ .

Let  $\pi_1, \pi_2 : \mathbb{R}^n \rightarrow \mathbb{S}^n$  be stereographic projections relative to the “north” and “south” poles on  $\mathbb{S}^n$ , respectively, so  $\pi_2^{-1} \circ \pi_1 = \text{id}^* = (\text{id}^*)^{-1}$ . Observe that

$$((\text{id}^*)^{-1} \circ \varphi^*)(x) = \frac{\varphi^*(x)}{|\varphi^*(x)|^2} = |x| \varphi\left(\frac{x}{|x|}\right) = \bar{\varphi}(x)$$

holds for all  $x \in \mathbb{R}^n \setminus \{0\}$ . It follows that

$$x \mapsto \begin{cases} (\pi_1^{-1} \circ \psi_1)(x), & \text{if } x \in M_1 \\ (\pi_2^{-1} \circ \Phi \circ \psi_2)(x), & \text{if } x \in M_2 \end{cases}$$

is a  $C^1$ -diffeomorphism of  $M_\varphi^n$  onto  $\mathbb{S}^n$ .  $\square$

By [Hir94, Thm 2.2.10], if two  $C^\infty$ -smooth manifolds are  $C^1$ -diffeomorphic, then they are  $C^\infty$ -diffeomorphic. It follows that there exist  $C^1$ -diffeomorphisms of collars in  $\mathbb{R}^7$  that do not admit diffeomorphic extensions of class  $LW_2^p$ , for any  $p > 7$ .

The next result follows from the inclusion  $W_{loc}^{2,p}(\Omega; \Omega') \subseteq W_{loc}^{2,q}(\Omega; \Omega')$ , for  $q \leq p$ .

**Corollary 1.7.** *Let  $n = 7$ . For  $p > n$ , there exist collars  $\Omega, \Omega'$  in  $\mathbb{R}^n$  and homeomorphisms  $\varphi : \Omega \rightarrow \Omega'$  of class  $LW_2^p$  that admit homeomorphic extensions of class  $LW_2^q$ , for every  $1 \leq q < n$ , but not of class  $LW_2^p$ .*

Since the above discussion relies crucially on Sobolev embedding theorems, it leaves open the borderline case  $p = n$ .

**Question 1.8.** Is Theorem 1.3 true for the case  $p = n$ ?

For  $p > n$ , the main obstruction to an extension theorem is the existence of exotic  $n$ -spheres. It is known that no exotic spheres exist for  $n = 1, 2, 3, 5, 6$  [KM63], and the case  $n = 1$  can be done by hand. It would be interesting to determine whether other geometric obstructions arise.

**Question 1.9.** For  $n = 2, 3, 5, 6$ , is Theorem 1.3 true for all  $p \geq 1$ ?

The outline of the paper is as follows. In Section 2 we review basic facts about Lipschitz mappings, Sobolev spaces, and the class  $LW_2^p$ . In Section 3 we prove extension theorems in the setting of doubly-punctured domains. Section 4 addresses the case of homeomorphisms between collars, by employing suitable generalizations of inversion maps and reducing to previous cases.

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## 2. NOTATION AND BASIC FACTS

For  $A \subset \mathbb{R}^n$ , we write  $A^c$  for the complement of  $A$  in  $\mathbb{R}^n$ . The open unit ball in  $\mathbb{R}^n$  is denoted  $\mathbb{B}^n$ ; if the dimension is understood, we will write  $\mathbb{B}$  for  $\mathbb{B}^n$ .

We write  $A \lesssim B$  for inequalities of the form  $A \leq kB$ , where  $k$  is a fixed dimensional constant and does not depend on  $A$  or  $B$ .

For domains  $\Omega$  and  $\Omega'$  in  $\mathbb{R}^n$ , recall that a map  $f : \Omega \rightarrow \Omega'$  is *Lipschitz* whenever

$$L(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in \Omega, x \neq y \right\} < \infty.$$

The map  $f$  is *locally Lipschitz* if every point in  $\Omega$  has a neighborhood on which  $f$  is Lipschitz. A homeomorphism  $f : \Omega \rightarrow \Omega'$  is *bi-Lipschitz* (resp. *locally bi-Lipschitz*) if  $f$  and  $f^{-1}$  are both Lipschitz (resp. locally Lipschitz); compare Equation (1.1).

The following lemmas about bi-Lipschitz maps are used in Section 2. The first is a special case of [TV81, Lemma 2.17]; the second one is elementary, so we omit the proof.

**Lemma 2.1** (Tukia-Väisälä). *Let  $O$  and  $O'$  be open, connected sets in  $\mathbb{R}^n$  and let  $K$  be a compact subset of  $O$ . If  $f : O \rightarrow O'$  is locally bi-Lipschitz, then  $f|_K$  is bi-Lipschitz, where  $L((f|_K)^{-1})$  depends only on  $O$ ,  $K$ , and  $L(f)$ .*

**Lemma 2.2.** *For  $i = 1, 2$ , let  $h_i : \Omega_i \rightarrow \mathbb{R}^n$  be locally bi-Lipschitz embeddings so that  $h_1(\Omega_1 \setminus \Omega_2) \cap h_2(\Omega_2 \setminus \Omega_1) = \emptyset$ . If  $h_1 = h_2$  holds on all of  $\Omega_1 \cap \Omega_2$ , then*

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in \Omega_1 \\ h_2(x), & \text{if } x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

*is also a locally bi-Lipschitz embedding.*

For  $f \in W^{2,p}(\Omega; \Omega')$ , we will use the Hilbert-Schmidt norm for the weak derivatives  $Df(x) := [\partial_j f_i(x)]_{i,j=1}^n$  and  $D^2 f(x) := [\partial_k \partial_j f_i(x)]_{i,j,k=1}^n$ . That is,

$$|Df(x)| := \left[ \sum_{i,j=1}^n |\partial_j f_i(x)|^2 \right]^{1/2}, \quad |D^2 f(x)| := \left[ \sum_{i,j,k=1}^n |\partial_k \partial_j f_i(x)|^2 \right]^{1/2}.$$

In what follows, we will use basic facts about Sobolev spaces, such as the change of variables formula [Zie89, Thm 2.2.2] and that Lipschitz functions on  $\Omega$  are characterized by the class  $W^{1,\infty}(\Omega)$  [EG92, Thm 4.2.3.5]. The lemma below gives a gluing procedure for Sobolev functions.

**Lemma 2.3.** *For  $i = 1, 2$ , let  $O_i$  be a domain in  $\mathbb{R}^n$  and let  $f_i \in W_{loc}^{1,p}(O_i)$ . If  $f_1 = f_2$  holds a.e. on  $O_1 \cap O_2$ , then  $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2 \in W_{loc}^{1,p}(O_1 \cup O_2)$ .*

*Proof.* Let  $O$  be a bounded domain in  $\mathbb{R}^n$  so that  $\bar{O} \subset O_1 \cup O_2$ . For each  $x \in O$ , there exists  $r > 0$  so that  $B(x, r)$  lies entirely in  $O_1$  or in  $O_2$ . Since  $\bar{O}$  is compact, there exists  $N \in \mathbb{N}$  and a collection of balls  $\{B(x_i, r_i)\}_{i=1}^N$  whose union covers  $O$ .

Let  $\{\varphi_i\}_{i=1}^N$  be a smooth partition of unity that is subordinate to the cover  $\{B(x_i, r_i)\}_{i=1}^N$ . For each  $i = 1, 2, \dots, N$ , one of  $f_1\varphi_i$  or  $f_2\varphi_i$  is well-defined and lies in  $W^{1,p}(O)$ ; call it  $\psi_i$ . We now observe that  $\psi := \sum_{i=1}^N \psi_i$  also lies in  $W^{1,p}(O)$  and by construction, it agrees with  $\chi_{O_1}f_1 + \chi_{O_2 \setminus O_1}f_2$ .  $\square$

It is a fact that the class  $LW_2^p$  is preserved under composition. This is stated as a lemma below, and it follows directly from the product rule [EG92, Thm 4.2.2.4] and the change of variables formula [Zie89, Thm 2.2.2].

**Lemma 2.4.** *Let  $p \geq 1$ . If  $f : \Omega \rightarrow \Omega'$  and  $g : \Omega' \rightarrow \Omega''$  are homeomorphisms of class  $LW_2^p$ , then so is  $h := g \circ f$ . In addition, for a.e.  $x \in \Omega$  and for all  $i, j, k \in \{1, \dots, n\}$ , the weak derivatives satisfy*

$$(2.1) \quad \begin{cases} \partial_j h_i(x) = \sum_{l=1}^n \partial_l g_i(f(x)) \partial_j f_l(x) \\ \partial_{kj}^2 h_i(x) = \sum_{l=1}^n \left[ \partial_l g_i(f(x)) \partial_{kj}^2 f_l(x) + \sum_{m=1}^n \partial_{ml}^2 g_i(f(x)) \partial_k f_m(x) \partial_j f_l(x) \right]. \end{cases}$$

**Remark 2.5.** Linear maps (homeomorphisms) such as dilation and translation, are clearly of class  $LW_2^p$ . So if  $g : \Omega \rightarrow \Omega'$  is any homeomorphism of class  $LW_2^p$ , then by Lemma 2.4, its composition with such linear maps is also of class  $LW_2^p$ . In what follows, we will implicitly use this fact to obtain convenient geometrical configurations.

### 3. EXTENSIONS FOR HOMEOMORPHISMS OF CLASS $LW_2^p$ BETWEEN DOUBLY-PUNCTURED DOMAINS

First we formulate the extension theorem in a different geometric configuration.

**Theorem 3.1.** *Let  $p \geq 1$ , let  $E_1$  and  $E_2$  be Jordan domains so that  $\overline{E_1} \cap \overline{E_2} = \emptyset$ , and let  $B_1$  and  $B_2$  be balls so that  $\overline{B_1} \cap \overline{B_2} = \emptyset$ .*

*If  $g : (E_2 \cup E_1)^c \rightarrow (B_1 \cup B_2)^c$  is a homeomorphism of class  $LW_2^p$  so that  $g(\partial E_i) = \partial B_i$  holds, for  $i = 1, 2$ , then there exists a homeomorphism  $G : E_2^c \rightarrow B_2^c$  of class  $LW_2^p$  and a neighborhood  $N$  of  $\partial E_2$  so that  $g|(N \cap E_2^c) = G|(N \cap E_2^c)$ .*

Following the outline of [Geh67, Sect 3], we begin with a special case.

**Lemma 3.2.** *Theorem 3.1 holds under the additional assumption that*

$$(3.1) \quad g|_{B^c} = \text{id}|_{B^c}$$

where  $B$  is an open ball that contains  $\bar{E}_1$  and  $\bar{E}_2$ .

*Proof. Step 1.* By composing with linear maps, we may assume that  $B = \mathbb{B}$ , and that there exist  $a, b \in \mathbb{R}$  so that  $a < b$  and  $\bar{B}_1 \subset \{x_n < a\}$  and  $\bar{B}_2 \subset \{x_n > b\}$ .

Put  $c = (b - a)/2$ . Define an odd,  $C^{1,1}$ -smooth function  $s_0 : \mathbb{R} \rightarrow [-1, 1]$  by

$$s_0(t) := \begin{cases} 1 - (t - c)^2/c^2, & \text{if } 0 \leq t \leq c \\ 1, & \text{if } t > c \end{cases}$$

and using the auxiliary function  $s : \mathbb{R} \rightarrow [0, 3]$ , given by

$$s(t) := \frac{3}{2} \left( s_0 \left( t - \frac{a+b}{2} \right) + 1 \right)$$

we define a bi-Lipschitz homeomorphism  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$(3.2) \quad S(x) = x - s(x_n) e_1.$$

It is clear that  $S$  is of class  $LW^p$  and satisfies the a.e. estimate

$$(3.3) \quad |D^2 S| \leq 2c^{-2}.$$

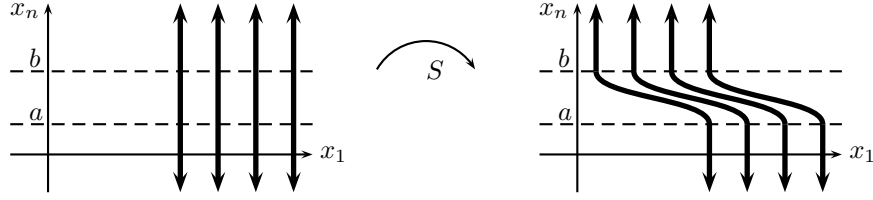


FIGURE 1. For  $\mathbb{R}^2$ , level curves for the map  $S$ .

*Step 2.* For  $k \in \mathbb{Z}$ , put  $\tau_k(x) = x + 3ke_1$  and consider the sets

$$\Omega := \left( \bigcup_{k=0}^{\infty} \tau_k(E_1) \cup \tau_k(E_2) \right)^c \text{ and } \Omega' := \left( \bigcup_{k=0}^{\infty} \tau_k(B_1) \cup \tau_k(B_2) \right)^c.$$

We now modify  $g$  into a new homeomorphism  $g_* : \Omega \rightarrow \Omega'$ , as follows:

$$(3.4) \quad g_*(x) := \begin{cases} (\tau_k \circ g \circ \tau_{-k})(x), & \text{if } x \in \Omega \cap \tau_k(\mathbb{B}), \text{ for some } k \geq 0 \\ x, & \text{if } x \in \Omega \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B}). \end{cases}$$

By our hypotheses, there exists  $r \in (0, 1)$  so that  $E_1 \cup E_2 \subset B(0, r)$  and so that  $g|_{\mathbb{B} \setminus B(0, r)} = \text{id}$ . Putting  $\Omega_1 := \tau_k(\mathbb{B}) \cap \Omega$  and  $\Omega_2 := \Omega \setminus \bigcup_{l=0}^{\infty} \tau_l(\overline{B(0, r)})$  for each  $k \in \mathbb{N}$ , Lemma 2.2 implies that  $g_*$  is locally bi-Lipschitz.

Similarly, for any bounded domain  $O$  in  $\Omega$  that meets  $\tau_k(\partial\mathbb{B})$ , put  $O_1 := O \cap \Omega$  and  $O_2 := O \setminus \tau_k(\overline{B(0, r)})$ . For  $f_1 := D(\tau_k \circ g \circ \tau_{-k})$  and  $f_2 := D(\text{id})$ , Lemma 2.3 implies that  $g_* \in W^{2,p}(O)$  and therefore  $g_* \in W_{loc}^{2,p}(\Omega; \Omega')$ . By symmetry, the same is true of  $g_*^{-1}$ , so  $g_*$  is of class  $LW_2^p$ .

*Step 3.* Consider the bi-Lipschitz homeomorphism given by

$$(3.5) \quad G_* := \tau_1 \circ g_*^{-1} \circ S \circ g_*.$$

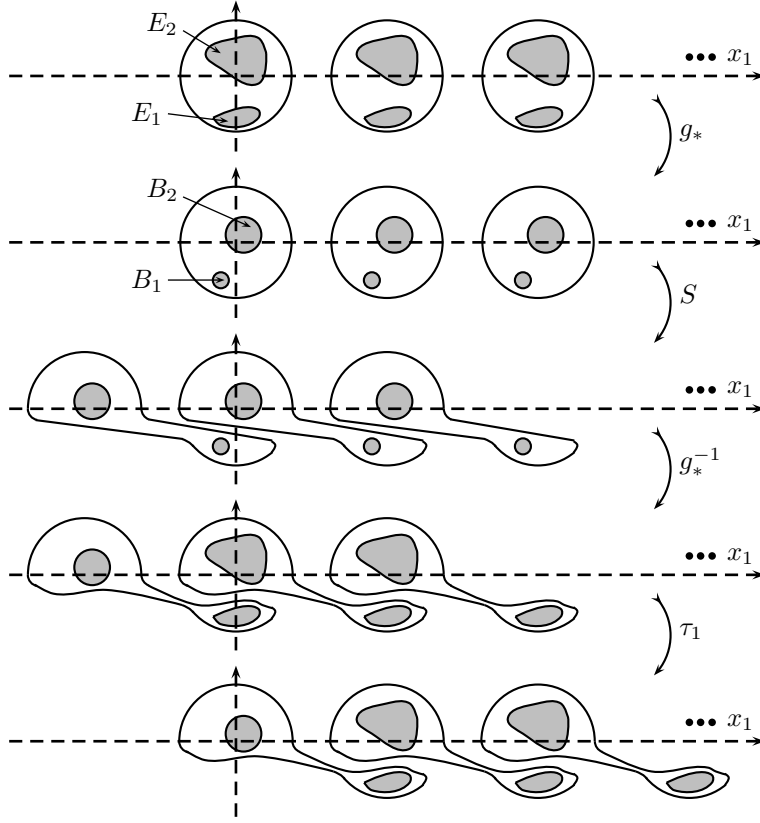
By Lemma 2.4, it is also of class  $LW_2^p$ . We now define  $G : E_2^c \rightarrow B_2^c$  as

$$(3.6) \quad G(x) := \begin{cases} G_*(x), & \text{if } x \in \Omega \\ \tau_1(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\ x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases}$$

By the same argument as [Geh67, pp. 153-4], the map  $G$  is a homeomorphism. We also note that  $G$  is “periodic” in the sense that, for each  $k \in \mathbb{N}$ ,

$$(3.7) \quad (\tau_k \circ G \circ \tau_{-k})|_{\tau_k(\mathbb{B} \setminus E_2)} = G|_{\tau_k(\mathbb{B} \setminus E_2)}.$$

To see that  $G$  extends  $g$ , consider the set  $\sigma_{ab} := g_*^{-1}(\{a \leq x_n \leq b\})$ . Its complement  $\mathbb{R}^n \setminus \sigma_{ab}$  consists of two (connected) components. Let  $\sigma_b$  be the component containing the vector  $e_n$ , let  $\sigma_a$  be the component containing  $-e_n$ , and consider

FIGURE 2. A schematic of the mapping  $G_*$ .

the open set  $N := \mathbb{B} \cap \sigma_b$ . By assumption,  $\bar{B}_2$  lies in  $\mathbb{B} \cap \{x_n > b\}$ , so  $\bar{E}_2$  lies in  $N$ . From before, we have  $g_* = g$  on  $\mathbb{B}$  and  $S = \tau_{-1}$  on  $\{x_n > b\}$ , which imply that

$$(S \circ g_*)(N) = (\tau_{-1} \circ g)(N) = \tau_{-1}(\mathbb{B} \cap \{x_n > b\}) \subset \tau_{-1}(\mathbb{B}).$$

By hypothesis we have  $g_*^{-1} = \text{id}$  on  $\tau_{-1}(\mathbb{B})$  and hence on  $(S \circ g)(N)$ . It follows that

$$G|N = G_*|N = (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|N = (\tau_1 \circ \text{id} \circ \tau_{-1} \circ g)|N = g|N.$$

As a result,  $G$  agrees with  $g$  on  $N \cap E_2^c$ .

Lastly,  $G = \text{id}$  holds on  $\sigma_b \setminus E_2$  and  $G = \tau_1$  holds on  $\sigma_a$ . Using these domains for  $\Omega_1$  and  $\mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B})$  for  $\Omega_2$ , Lemma 2.2 implies that  $G$  is locally bi-Lipschitz. With the same choice of domains, Lemma 2.3 further implies that  $G \in W_{loc}^{2,p}(E_2^c; B_2^c)$ . For the case of  $G^{-1}$ , note that the inverse is given by

$$(3.8) \quad G^{-1}(x) = \begin{cases} G_*^{-1}(x), & \text{if } x \in \Omega \setminus \tau_{-1}(B_2) \\ \tau_{-1}(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\ x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases}$$

Arguing similarly with  $g_*(N)$  for  $N$ , it follows that  $G^{-1} \in W_{loc}^{2,p}(B_2^c; E_2^c)$ , which proves the lemma.  $\square$

We now observe that Lemma 3.2 holds true even when  $B_1$  and  $B_2$  are not balls. In the preceding proof it is enough that, up to rotation, there is a slab  $\{c_1 < x_n < c_2\}$  that separates  $B_1$  from  $B_2$ . This result, stated below, is used in Section 4.

**Lemma 3.3.** *Let  $p \geq 1$  and let  $E_1$ ,  $E_2$ ,  $C_1$ , and  $C_2$  be Jordan domains so that  $\overline{E_1} \cap \overline{E_2} = \emptyset$  and  $\overline{C_1} \cap \overline{C_2} = \emptyset$ . If  $g : (E_1 \cup E_2)^c \rightarrow (C_1 \cup C_2)^c$  is a homeomorphism of class  $LW_2^p$  so that*

- (1)  $g(\partial E_i) = \partial B_i$  holds, for  $i = 1, 2$ ,
- (2) there exists a ball  $B$  containing  $\bar{E}_1$  and  $\bar{E}_2$  so that  $g|_{B^c} = \text{id}|_{B^c}$ ,
- (3) there exist a rotation  $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and numbers  $c_1, c_2 \in \mathbb{R}$ , with  $c_1 < c_2$ , so that  $\Theta(C_1) \subset \{x_n < c_1\}$  and  $\Theta(C_2) \subset \{x_n > c_2\}$ ,

*then there is a homeomorphism  $G : E_2^c \rightarrow C_2^c$  of class  $LW_2^p$  and a neighborhood  $N$  of  $\partial E_2$  so that  $g|(N \cap E_2^c) = G|(N \cap E_2^c)$ .*

Though the regularity of the extension  $G$  is local in nature, it nonetheless enjoys certain uniform properties. We summarize them in the next lemma.

**Lemma 3.4.** *Let  $E_1$ ,  $E_2$ ,  $C_1$ ,  $C_2$ ,  $B$ , and  $g$  be as in Lemma 3.3. If  $G$  is the extension of  $g$  as defined in Equation (3.6), then*

- (1)  $DG \in L^\infty(E_2^c)$  and  $DG^{-1} \in L^\infty(C_2^c)$ ;
- (2) the restriction  $G|_{B^c}$  is a bi-Lipschitz homeomorphism.

*Proof.* From Lemma 3.3, the map  $G$  is already locally bi-Lipschitz. To prove item (1), we will give a uniform bound for  $L(G|_K)$  over all compact subsets  $K$  of  $B^c$ . Let  $B = \mathbb{B}$  and let  $S$  and  $g_*$  be as defined in the proof of Lemma 3.2.

Again, let  $\sigma_{ab} := g_*^{-1}(\{a \leq x_n \leq b\})$  and let  $\sigma_b$  and  $\sigma_a$  be the (connected) components of  $\mathbb{R}^n \setminus \sigma_{ab}$  containing the vectors  $e_n$  and  $-e_n$ , respectively. By Equation (3.2), we have  $S|\{x_n < a\} = \text{id}$  and  $S|\{x_n > b\} = \tau_{-1}$ , which imply, respectively, the bounds  $L(G|\mathbb{B}^c \cap \sigma_a) \leq 1$  and  $L(G|\mathbb{B}^c \cap \sigma_b) \leq 1$ .

It remains to estimate  $L(G|\mathbb{B}^c \cap \sigma_{ab})$ . For each  $k \in \mathbb{N}$ , the set  $\sigma_{ab}^k := \sigma_{ab} \cap \tau_k(\mathbb{B})$  is compact, so by Lemma 2.1, the restriction  $G|_{\sigma_{ab}^k}$  is bi-Lipschitz. Equation (3.7) then implies that  $L(G|\sigma_{ab}^k) = L(G|\sigma_{ab}^1)$  holds for each  $k \in \mathbb{N}$ .

The remaining set  $\sigma_{ab} \setminus \bigcup_{k=0}^\infty \tau_k(\mathbb{B})$  consists of infinitely many components, one of which is an unbounded subset  $U$  of  $\{x_1 < 0\}$  and the others are translates of a compact subset  $K_0$  of  $\sigma_{ab} \cap B(0, 3)$ . Since  $g|_U = \text{id}$ , it follows that

$$G|_\sigma = (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|_\sigma = (\tau_1 \circ S)|_\sigma$$

from which  $L(G|\sigma) \leq L(S)$  follows. By the ‘periodicity’ of  $G$  (Equation (3.7)), for all  $k \in \mathbb{N}$  we also have  $L(G|\tau_k(K_0)) = L(G|K_0)$ . Item (1) of the lemma follows from [EG92, Thm 4.2.3.5] and from the above estimates, where

$$\|DG\|_{L^\infty(E_2^c)} \leq \max\{1, L(G|K_0), L(G|\sigma_{ab}^1), L(S)\}.$$

Using the explicit formula in Equation (3.8), the case of  $G^{-1}$  follows similarly.

To prove item (2), let  $\ell$  be any line segment that does not intersect  $\mathbb{B}$ . The restriction  $G|_\ell$  is bi-Lipschitz with  $L(G|_\ell) \leq C$ . Since  $\partial\mathbb{B}$  is compact, it follows from Lemma 2.1 that the restriction  $G|_{\partial\mathbb{B}}$  is bi-Lipschitz.

Let  $x_1$  and  $x_2$  be arbitrary points in  $\mathbb{B}^c$  and let  $\ell$  be the line segment in  $\mathbb{R}^n$  which joins  $x_1$  to  $x_2$ . If  $\ell$  crosses through  $\mathbb{B}$ , then let  $y_1$  and  $y_2$  be points on  $\ell \cap \partial\mathbb{B}$ , where  $|x_1 - y_1| < |x_1 - y_2|$ . Since  $\ell$  is a geodesic, we have the identity

$$|x_1 - x_2| = |x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|.$$



The Triangle inequality then implies that

$$\begin{aligned}
|G(x_1) - G(x_2)| &\leq |G(x_1) - G(y_1)| + |G(y_1) - G(y_2)| + |G(y_2) - G(x_2)| \\
&\leq C(|x_1 - y_1| + |y_2 - x_2|) + L(G|\partial\mathbb{B})|y_1 - y_2| \\
&\leq (C + L(G|\partial\mathbb{B}))(|x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|) \\
&= (C + L(G|\partial\mathbb{B}))|x_1 - x_2|.
\end{aligned}$$

Again, the argument is symmetric for  $G^{-1}$ , so this proves the lemma.  $\square$

Theorem 3.1 now follows easily from Lemma 3.2, and a more general version of the theorem follows from Lemma 3.3. As in [Geh67, Lemma 2], one takes compositions with the extension, its inverse, and a radial stretch map.

*Proof of Theorem 3.1.* By composing  $g$  with linear maps, we may assume that  $E_1, E_2, B_1$  and  $B_2$  are subsets of  $\mathbb{B}$ , that  $0 \in E_2$ , and that  $\mathbb{B}^c \subset g(\mathbb{B}^c)$ . Choose  $r_1, r_2 \in (0, 1)$  so that  $B(0, r_1) \subset E_2$  and that  $E_1 \cup E_2 \subset B(0, r_2)$ .

Let  $\rho : [0, \infty) \rightarrow [0, \infty)$  be a smooth increasing function so that  $\rho([0, r_1]) = [0, r_2]$  and  $\rho([1, \infty)) = [1, \infty)$ . Define a homeomorphism  $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$(3.9) \quad R(x) := \begin{cases} \rho(|x|) \cdot |x|^{-1}x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly,  $R$  is of class  $LW_2^p$  and bi-Lipschitz, and maps  $B(0, r_1)$  onto  $B(0, r_2)$ .

Putting  $E'_1 := (g \circ R)(E_1)$  and  $E'_2 := ((g \circ R)(E_2^c))^c$ , Lemma 2.4 implies that

$$h := g \circ R \circ g^{-1} : (E'_1 \cup E'_2)^c \rightarrow (B_1 \cup B_2)^c$$

is also a homeomorphism of class  $LW_2^p$ . Since  $R|\mathbb{B}^c = \text{id}|\mathbb{B}^c$ , we further obtain

$$(3.10) \quad h|\mathbb{B}^c = (g \circ R \circ g^{-1})|\mathbb{B}^c = \text{id}|\mathbb{B}^c.$$

So with  $E'_1$  and  $E'_2$  in place of  $E_1$  and  $E_2$ , respectively,  $h$  satisfies Equation (3.1) and the other hypotheses of Lemma 3.2. As a result, there exists a homeomorphism  $H : (E'_2)^c \rightarrow B_2^c$  of class  $LW_2^p$  and a neighborhood  $N'$  of  $\partial E'_2$  so that

$$h|(N' \cap (E'_2)^c) = H|(N' \cap (E'_2)^c).$$

Let  $G := H \circ g \circ R^{-1}$ . The open set

$$N := (R \circ g^{-1})(N' \setminus (\bar{B}_1 \cup \bar{B}_2))$$

contains  $\partial E_2$ , and by Lemma 2.4, the map  $G$  is of class  $LW_2^p$ . Moreover, for each  $x \in N \setminus E_2$ , there is a  $y \in N' \setminus D'_2$  so that  $x = (R \circ g^{-1})(y)$  and therefore

$$\begin{aligned}
G(x) &= (H \circ g \circ R^{-1})((R \circ g^{-1})(y)) = H(y) \\
&= h(y) = (g \circ R \circ g^{-1})((g \circ R^{-1})(x)) = g(x).
\end{aligned}$$

We thereby obtain  $g = G$  on  $N \cap E_2^c$ , as desired.  $\square$

#### 4. EXTENSIONS OF HOMEOMORPHISMS OF CLASS $LW_2^p$ BETWEEN COLLARS

**4.1. Generalized Inversions.** To pass to the configurations of domains in Theorem 1.3, we will use *generalized inversions*. For fixed  $a, r > 0$ , these are homeomorphisms  $I_{a,r} : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$  of the form

$$I_{a,r}(x) := r^{a+1}|x|^{-(a+1)}x.$$

Indeed, the inverse map satisfies  $(I_{a,r})^{-1} = I_{1/a,r}$ , as well as the estimate

$$(4.1) \quad |x|^{a+1} = (r^{1/a+1} |I_{a,r}(x)|^{-1/a})^{a+1} \approx |I_{a,r}(x)|^{-(1/a+1)}.$$

For derivatives of  $I_{a,r}$ , an elementary computation gives

$$(4.2) \quad |D^k I_{a,r}(x)| \lesssim r^{a+1} |x|^{-(a+k)}$$

and similarly, for the Jacobian determinant  $JI_{a,r} := |\det(DI_{a,r})|$  we have

$$(4.3) \quad JI_{a,r}(x) \leq n r^{n(a+1)} |x|^{-n(a+1)} \approx |I_{a,r}(x)|^{n(a+1)/a}.$$

If  $a = 1$ , then  $I_{1,r}$  is conformal and maps spheres to spheres. In general, the map  $I_{a,r}$  possesses weaker properties which are sufficient for our purposes. For instance, it preserves radial rays, or sets of the form  $\{\lambda x : \lambda > 0\}$  for some  $x \in \mathbb{R}^n \setminus \{0\}$ .

Another property, stated below, is used in the proof of Theorem 1.3 under the following hypotheses. To begin, write  $B_1 = B(t, r_1)$  and  $B_2 = B(z, r_2)$ , where  $\bar{B}_1 \subset B_2$ . By composing with linear maps, we may assume that

- (H1) The  $x_n$ -coordinate axis crosses through the points  $t$  and  $z$ , with  $t_n \leq z_n \leq 0$ . As a result, the ‘south poles’  $\tau := t - r_1 \vec{e}_n$  on  $\bar{B}_1$  and  $\zeta := z - r_2 \vec{e}_n$  on  $\bar{B}_2$  satisfy  $\zeta_n < \tau_n$  and  $|\zeta - \tau| = \text{dist}(\bar{B}_1, B_2^c)$ .
- (H2) There exists  $r \in (0, r_2)$  so that the sphere  $\partial B(0, r)$  is tangent to both  $\partial B_1$  and  $\partial B_2$ , with  $B(0, r) \subset B_2 \setminus B_1$ . In particular, this gives  $r_1 < |t_n|$ .

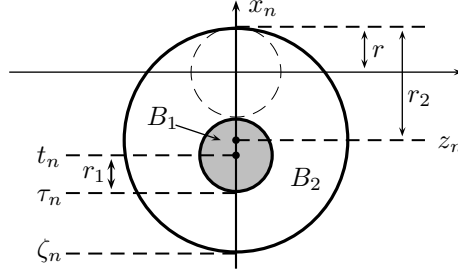


FIGURE 3. A possible configuration for  $B_1$ ,  $B_2$ , and  $B(0, r)$ .

**Lemma 4.1.** *Let  $a \in (0, 1)$ . If  $B_1$  and  $B_2$  are balls in  $\mathbb{R}^n$  with  $\bar{B}_1 \subset B_2$  and which satisfy hypotheses (H1) and (H2), then there exist real numbers  $c_1 < c_2$  so that  $I_{a,r}(B_1) \subset \{x_n < c_1\}$  and  $I_{a,r}(B_2^c) \subset \{x_n > c_2\}$ .*

The proof is a computation, and the basic idea is simple. Though the bounded domains  $I_{a,r}(B_1)$  and  $I_{a,r}(B_2^c)$  may not be balls, the distance between them is still attained by the images of the ‘north’ and ‘south’ poles of  $B_1$  and  $B_2$ , respectively.

*Proof.* Once again, let  $\tau$  and  $\zeta$  be the “south poles” of  $B_1$  and  $B_2$ , respectively. From Hypotheses (H1) and (H2), we have

$$\zeta_n = -|\zeta| < -|\tau| = \tau_n.$$

and putting  $I = I_{a,r}$ , the image points  $\tau' := I(\tau)$  and  $\zeta' := I(\zeta)$  therefore satisfy

$$(4.4) \quad \tau'_n = -|\tau'| < -|\zeta'| = \zeta'_n.$$

**Claim 4.2.** For all  $y' \in I(B_1)$ , we have  $y'_n < \tau'_n$ .

Supposing otherwise, there exists  $y \in \partial B_1$  with  $y \neq \tau$  and so that  $y'$  has the same  $n$ th coordinate as  $\tau'$ . Let  $\theta$  be the angle between the  $x_n$ -axis and the line crossing through  $y'$  and 0. By our hypotheses, we have  $t_n \leq 0$  and  $0 < \theta < \frac{\pi}{2}$  and therefore  $0 < \cos \theta < 1$ . From  $|\tau| = r_1 - t_n$ , we obtain

$$|y'| = \frac{|\tau'|}{\cos \theta} = \frac{r^{a+1}|\tau|^{-a}}{\cos \theta} = \frac{r^{a+1}}{(r_1 - t_n)^a \cos \theta}$$

so from  $|y'| = r^{a+1}|y|^{-a}$  and the above identity, we further obtain

$$(4.5) \quad |y| = r^{(a+1)/a} \left[ \frac{r^{a+1}}{(r_1 - t_n)^a \cos \theta} \right]^{-1/a} = (\cos \theta)^{1/a} (r_1 - t_n).$$

On the other hand,  $I$  preserves radial rays and hence angles between radial rays. As a result,  $y \in \partial B_1$  (and the Law of Cosines) imply that

$$\begin{aligned} r_1^2 &= |y|^2 + t_n^2 - 2|y|t_n \cos \theta, \\ \text{so } |y| &= -t_n \cos \theta + \sqrt{r_1^2 - t_n^2 \sin^2 \theta}. \end{aligned}$$

From Hypothesis (H2) once again, we obtain  $r_1 < |\tau_n|$  and hence

$$|y| < -t_n \cos \theta + \sqrt{r_1^2 - r_1^2 \sin^2 \theta} = (r_1 - t_n) \cos \theta.$$

This is in contradiction with Equation (4.5), since the inequality  $\cos \theta \leq (\cos \theta)^{1/a}$  follows from  $a \geq 1$ . The claim follows.

**Claim 4.3.** For all  $w' \in I(B_2^c)$ , we have  $\zeta'_n < w'_n$ .

Suppose there exists  $w \in \partial B_2$  so that  $w \neq \zeta$  and  $w'_n = \zeta'_n$ . If  $\alpha$  is the angle between  $w$  and the  $x_n$ -axis, then a similar computation as above gives

$$(2r_2 - r) \cos^{1/a} \alpha = |w| = (r_2 - r) \cos \alpha + \sqrt{r_2^2 - (r_2 - r)^2 \sin^2 \alpha}$$

Computing further, we obtain  $\psi(a) = r_2^2$ , where  $\psi : (0, \infty) \rightarrow (0, \infty)$  is given by

$$\psi(a) := ((2r_2 - r) \cos^{1/a} \alpha - (r_2 - r) \cos \alpha)^2 + (r_2 - r)^2 \sin^2 \alpha$$

Clearly  $\psi$  is smooth and an elementary computation shows that it attains a minimum at a unique point in  $(0, 1)$ . We observe that

$$\psi(1) = r_2^2 \cos^2 \alpha + (r_2 - r)^2 \sin^2 \alpha < r_2^2.$$

Since  $0 < \cos \alpha < 1$ , we see that  $\cos^{1/a} \alpha \rightarrow 0$  as  $a \rightarrow 0$ . It follows that

$$\lim_{a \rightarrow 0} \psi(a) = (0 + (r_2 - r) \cos \alpha)^2 + (r_2 - r)^2 \sin^2 \alpha = (r_2 - r)^2 < r_2^2$$

and therefore  $\psi(a) < r_2^2$  holds for all  $(0, 1)$ . This is a contradiction, which proves Claim 4.3. Combining both claims and Equation (4.4), the lemma follows.  $\square$

**4.2. From Doubly-Punctured Domains to Collars.** We now prove Theorem 1.3. The argument requires several lemmas.

**Lemma 4.4.** Let  $a > 0$  and let  $D_1, D_2, B_1, B_2$ , and  $f$  be given as in Theorem 1.3. If there exists  $r > 0$  so that  $\bar{B}(0, r) \subset D_2 \setminus D_1$  and  $\bar{B}(0, r) \subset B_2 \setminus B_1$ , and if  $f(0) = 0$ , then  $I_{a,r} \circ f \circ I_{a,r}^{-1}$  is a homeomorphism of class  $LW_2^p$ .

*Proof.* Since  $\Omega := I_{a,r}(D_2 \setminus (\bar{D}_1 \cup \{0\}))$  and  $I_{a,r}(B_2 \setminus (\bar{B}_1 \cup \{0\}))$  lie in  $\mathbb{R}^n \setminus B(0, \epsilon)$ , for some  $\epsilon > 0$ , the restricted maps  $I_{a,r}^{-1}|_\Omega$  and  $I_{a,r}|_{\Omega'}$  are diffeomorphisms. By Lemma 2.4, it follows that  $g := I_{a,r} \circ f \circ I_{a,r}^{-1} : \Omega \rightarrow \Omega'$  is of class  $LW_2^p$ .  $\square$

**Lemma 4.5.** *Let  $E_1, E_2, C_1, C_2, B$ , and  $g$  be given as in Lemma 3.3, and let  $G$  be given as in Equation (3.6). If  $0 \in E_2$ , if  $0 \in C_2$ , and if there exists  $r > 0$  so that  $B = B(0, r)$ , then for each  $a > 0$ , the map*

$$F(x) := \begin{cases} (I^{-1} \circ G \circ I)(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

*is a locally bi-Lipschitz homeomorphism.*

*Proof.* Without loss of generality, let  $r = 1$  and put  $I = I_{a,r}$  and  $b = 1/a$ . By Equation (3.6), we have  $|G(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ , so  $F$  is a well-defined homeomorphism. For each  $\epsilon > 0$ , put  $B_\epsilon := B(0, \epsilon)$ . The restrictions  $I|_{B_\epsilon^c}$  and  $I^{-1}|_{B_\epsilon^c}$  are diffeomorphisms, so  $F|_{B_\epsilon^c}$  is already locally bi-Lipschitz for each  $\epsilon > 0$ .

To show that  $F|_{B_\epsilon}$  is bi-Lipschitz, recall that  $DG \in L^\infty(E_2^c)$  follows from Lemma 3.4. So from Equations (2.1), (4.1), and (4.2), it follows that, for a.e.  $x \in I^{-1}(E_2^c)$ ,

$$\begin{aligned} |DF(x)| &\leq |DI^{-1}((G \circ I)(x))| |DG(I(x))| |DI(x)| \\ &\lesssim \frac{\|DG\|_\infty}{|(G \circ I)(x)|^{b+1} |x|^{a+1}} \approx \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}}. \end{aligned}$$

Now fix  $y_0 \in E_2^c$ . Putting  $L := L(G^{-1}|_{B^c})$ , for all  $x \in B_\epsilon$  we have

$$|G(I(x)) - G(y_0)| \geq L^{-1}(|I(x) - y_0|) \geq L^{-1}(|I(x)| - |y_0|).$$

Applying the triangle inequality to the right-hand side, we obtain

$$|G(I(x))| \geq L^{-1}(|I(x)| - |y_0|) - |G(y_0)|$$

and taking reciprocals, we further obtain

$$(4.6) \quad \begin{cases} \frac{|I(x)|}{|(G \circ I)(x)|} \leq \frac{L|I(x)|}{|I(x)| - |y_0| - L|G(y_0)|} \\ \qquad \qquad \qquad = \frac{L r^{a+1}}{r^{a+1} - |x|^a |y_0| - |x|^a L |G(y_0)|} \rightarrow L \end{cases}$$

as  $x \rightarrow 0$ . Combining the previous estimates, for sufficiently small  $\epsilon > 0$

$$|DF(x)| \lesssim \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}} \lesssim (2L)^{b+1} \|DG\|_\infty < \infty$$

holds for a.e.  $x \in B_\epsilon$ , and therefore  $|DF| \in L_{\text{loc}}^\infty(I^{-1}(E_2^c))$ . By [EG92, Thm 4.2.3.5], it follows that  $F$  is locally Lipschitz on  $B(0, \epsilon)$ . By symmetry, the same holds for  $F^{-1}$ , so  $F$  is locally bi-Lipschitz on all of  $I^{-1}(E_2^c)$ .  $\square$

In the remaining proofs, we will require explicit forms of the extensions from Lemma 3.2 and from Theorem 3.1.

**Lemma 4.6.** *Let  $E_1, E_2, C_1, C_2, g$ , and  $B = B(0, r)$  be given as in Lemma 4.5, let  $G$  be given as in Equation (3.6), and let  $p \in [1, n)$ . If  $a < n/p - 1$ , then the homeomorphism  $I_{a,r}^{-1} \circ G \circ I_{a,r}$  is of class  $LW_2^p$ .*

*Proof.* For convenience, we reuse the notation from the proof of Lemma 4.5. As before,  $I|_{B_\epsilon^c}$  and  $I^{-1}|_{B_\epsilon^c}$  are diffeomorphisms, so by Lemma 2.4, the map  $F|_{B_\epsilon^c}$  is of class  $LW_2^p$ . It suffices to show that  $F \in W_{loc}^{2,p}(B_\epsilon; \mathbb{R}^n)$  and  $F^{-1} \in W_{loc}^{2,p}(F(B_\epsilon); B_\epsilon)$ , for each  $\epsilon > 0$ .

To estimate second derivatives, we use Equations (2.1), (4.1), (4.2), and (4.6) once again. As a shorthand, put  $y := I(x)$  and  $z := (G \circ I)(x)$ . We then obtain

$$(4.7) \quad \left\{ \begin{aligned} |D^2 F(x)| &= |D^2(I^{-1} \circ G \circ I)(x)| \\ &\leq |D^2 I^{-1}(z)| |DG(y)|^2 |DI(x)|^2 \\ &\quad + |DI^{-1}(z)| \left( |D^2 G(y)| |DI(x)|^2 + |DG(y)| |D^2 I(x)| \right) \\ &\lesssim \frac{\|DG\|_\infty^2}{|z|^{b+2} |x|^{2(a+1)}} + \frac{1}{|z|^{b+1}} \left( \frac{|D^2 G(y)|}{|x|^{2(a+1)}} + \frac{\|DG\|_\infty}{|x|^{a+2}} \right) \\ &\lesssim \frac{|I(x)|^{2(b+1)}}{|G(I(x))|^{b+2}} + \frac{|I(x)|^{2(b+1)} |D^2 G(I(x))|}{|G(I(x))|^{b+1}} + \frac{|I(x)|^{b+1}}{|G(I(x))|^{b+1} |x|} \\ &\lesssim |I(x)|^b + |I(x)|^{b+1} |D^2 G(I(x))| + |x|^{-1} \end{aligned} \right.$$

for a.e.  $x \in B_\epsilon$ . Since  $p < n$  and  $b = 1/a$ , the function  $x \mapsto |I(x)|^b = |x|^{-1}$  lies in  $L^p(B_\epsilon)$ . For the remaining term, Equations (4.1) and (4.3) imply that

$$1 = JI^{-1}(I(x))JI(x) \lesssim |I^{-1}(I(x))|^{n(a+1)} JI(x) = |I(x)|^{-n(b+1)} JI(x)$$

so by a change of variables [Zie89, Thm 2.2.2] and Equation (4.3), we have

$$(4.8) \quad \left\{ \begin{aligned} \int_{B_\epsilon} |I(x)|^{p(b+1)} |D^2 G(I(x))|^p dx &\lesssim \int_{B_\epsilon} \frac{|D^2 G(I(x))|^p JI(x)}{|I(x)|^{(n-p)(b+1)}} dx \\ &= \int_{\mathbb{B}^c} \frac{|D^2 G(y)|^p}{|y|^{(n-p)(b+1)}} dy. \end{aligned} \right.$$

For each  $k \in \mathbb{N}$ , Equation (3.6) implies that  $G|_{\tau_k(E_2)} = \text{id}$  and  $G|_{\tau_k(E_1)} = \tau_1$ , and therefore  $D^2 G|_{\tau_k(E_1 \cup E_2)} = 0$ . The rightmost integral in Equation (4.8) can therefore be restricted to the subset

$$\Omega := \mathbb{B}^c \setminus \bigcup_{k=1}^{\infty} \tau_k(E_1 \cup E_2).$$

As defined in the proof of Lemma 3.2 the maps  $g_*$ ,  $G_*$ , and  $G$  satisfy

$$(4.9) \quad |D^2 G(y)| \lesssim |D^2 g_*^{-1}((S \circ g_*)(y))| + |D^2 S(g_*(y))| + |D^2 g_*(y)|$$

for a.e.  $y \in I^{-1}(E_2^c)$ , and where  $\lesssim$  includes the constants  $L(g_*)$ ,  $L(g_*^{-1})$ ,  $L(S)$ , and  $L(\tau_1)$ . Using the second derivative bound for  $S$  (Equation (3.3)), we obtain

$$\int_{\Omega} \frac{|D^2 S(g_*(y))|^p}{|y|^{(n-p)(b+1)}} dy \leq \int_{\Omega} \frac{2c^{2p}}{|y|^{(n-p)(b+1)}} dy \lesssim \int_1^\infty \frac{\rho^{n-1}}{\rho^{(n-p)(b+1)}} d\rho.$$

The rightmost integral is finite, since  $a < n/p - 1$  implies that  $b > p/(n-p)$  and

$$(n-1) - (n-p)(b+1) < (n-1) - (n-p)\left(\frac{p}{n-p} - 1\right) = -1.$$

For the other terms of Equation (4.9), Equation (3.4) implies that  $D^2 g_*^{-1}(z) = 0$  for a.e.  $z \notin \bigcup_{k=1}^{\infty} \tau_k(\mathbb{B})$ . Since  $S \circ g_*$  is locally bi-Lipschitz, we estimate

$$\begin{aligned} \int_{\Omega} \frac{|D^2 g_*^{-1}((S \circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}} dy &= \sum_{k=1}^{\infty} \int_{\tau_k((S \circ g_*)^{-1}(\mathbb{B})) \cap \Omega} \frac{|D^2 g_*^{-1}((S \circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}} dy \\ &\approx \sum_{k=1}^{\infty} \int_{g_*^{-1}(\Omega) \cap \tau_k(\mathbb{B})} \frac{|D^2 g_*^{-1}(z)|^p dz}{|(S \circ g_*)^{-1}(z)|^{(n-p)(b+1)}} \end{aligned}$$

Equation (3.2) implies that  $|S^{-1}(y)| \geq |y|$  holds, for each  $y \in \mathbb{R}^n$ , and therefore

$$|(S \circ g_*)^{-1}(z)| \geq 3k - 1 > k$$

holds, for each  $z \in \tau_k(\mathbb{B})$  and each  $k \in \mathbb{N}$ . From the above inequalities and another change of variables, we further estimate

$$\begin{aligned} \int_{g_*^{-1}(\Omega) \cap \tau_k(\mathbb{B})} \frac{|D^2 g_*^{-1}(z)|^p}{|(S \circ g_*)^{-1}(z)|^{(n-p)(b+1)}} dz &\lesssim \int_{g_*^{-1}(\Omega) \cap \tau_k(\mathbb{B})} \frac{|D^2 g_*^{-1}(z)|^p}{k^{(n-p)(b+1)}} dz \\ &\leq \frac{\int_{\mathbb{B} \setminus (C_1 \cup C_2)} |D^2 g^{-1}(z)|^p dz}{k^{(n-p)(b+1)}}, \\ \text{so } \int_{\Omega} \frac{|D^2 g_*^{-1}((S \circ g_*)(y))|^p}{|y|^{(n-p)(b+1)}} dy &\lesssim \sum_{k=1}^{\infty} \frac{\|D^2 g^{-1}\|_{L^p(\mathbb{B} \setminus (C_1 \cup C_2))}}{k^{(n-p)(b+1)}}. \end{aligned}$$

The rightmost sum is finite, since  $(n-p)(b+1) > 1$  follows from the hypothesis that  $a < n/p - 1$ . A similar estimate gives  $|y|^{(p-n)(b+1)} |D^2 g_*(y)| \in L^p(B_\epsilon)$ , so by Equations (4.7)-(4.9), we obtain  $|D^2 F| \in L^p(B_\epsilon)$ , as desired.

The same argument, with  $G^{-1}$  for  $G$ , shows that the map  $F^{-1} = I^{-1} \circ G^{-1} \circ I$  also lies in  $W_{loc}^{2,p}(F(B_\epsilon); B_\epsilon)$ . This proves the lemma.  $\square$

Using the previous lemmas, we now prove the main theorem.

*Proof of Theorem 1.3.* Let  $a < n/p - 1$  be given. By post-composing  $f$  with linear maps, we may assume that the balls  $B_1$  and  $B_2$  satisfy hypotheses (H1) and (H2) from Section 4.1, so in particular we have  $B(0, r) \subset B_2 \setminus \bar{B}_1$ . We further assume that  $B(0, r) \subset D_2 \setminus \bar{D}_1$  and  $f(0) = 0$ .

By Lemma 4.1, there exist  $c_1 < c_2$  so that  $B_1 \subset \{x_n < c_1\}$  and  $B_2 \subset \{x_n > c_2\}$ . For  $I := I_{a,r}$  and  $g := I \circ f \circ I^{-1}$ , Lemma 4.4 implies that  $g$  is of class  $LW_2^p$ .

Put  $E_1 = I(D_1)$ ,  $E_2 := I(D_2^c)^c$ ,  $C_1 := I(B_1)$ , and  $C_2 := I((B_2)^c)^c$ . By Lemma 3.3 and the proof of Theorem 3.1, there exists a homeomorphism  $G : E_2^c \rightarrow C_2^c$  of class  $LW_2^p$  and a neighborhood  $N'$  of  $\partial E_2$  so that

$$g|(N' \cap E_2^c) = G|(N' \cap E_2^c).$$

As a result, the homeomorphism  $F$ , as defined in Lemma 4.5, and the open set  $N := I^{-1}(N')$ , a neighborhood of  $\partial D_2$ , therefore satisfy the identity

$$f|(N \cap \bar{D}_2) = F|(N \cap \bar{D}_2).$$

Recalling the proof of Theorem 3.1, we have  $G = H \circ g \circ R^{-1}$ , where

- (H3)  $R$  is a diffeomorphism that agrees with the identity map on  $\mathbb{B}^c$ ;
- (H4)  $H$  is a homeomorphism of class  $LW_2^p$ , as given from Lemma 3.3, that agrees with  $h = g \circ R \circ g^{-1}$  on the open set  $(g \circ R)(N')$ .

Putting  $H_* := I^{-1} \circ H \circ I$  and  $R_* := I^{-1} \circ R \circ I$ , we rewrite

$$F = I^{-1} \circ (H \circ g \circ R^{-1}) \circ I = H_* \circ f \circ R_*^{-1}.$$

From property (H3) and properties of  $I$  and  $I^{-1}$ , we see that  $R_*^{-1}$  is a diffeomorphism from  $\mathbb{R}^n \setminus \{0\}$  onto itself. In particular, for each  $r > 0$  the restriction  $R_*^{-1}|_{B(0,r)^c}$  is bi-Lipschitz. On the other hand, for sufficiently small  $r > 0$  we have  $R^{-1} \circ I = I$  on  $B(0, r)$ . Letting  $\text{Id}_n$  be the  $n \times n$  identity matrix,

$$\begin{aligned} DR_*^{-1}|_{B(0,r)} &= D(I^{-1} \circ R^{-1} \circ I)|_{B(0,r)} = D(I^{-1} \circ I)|_{B(0,r)} = \text{Id}_n \\ D^2 R_*^{-1}|_{B(0,r)} &= D^2(I^{-1} \circ R^{-1} \circ I)|_{B(0,r)} = D^2(I^{-1} \circ I)|_{B(0,r)} = 0. \end{aligned}$$

This implies that  $R_*^{-1} \in W_{loc}^{2,p}(\mathbb{R}^n; \mathbb{R}^n)$  and by Lemma 2.2, that  $R_*^{-1}$  is bi-Lipschitz. By symmetry the same holds for  $R_* = I^{-1} \circ R \circ I$ , so  $R_*^{-1}$  is of class  $LW_2^p$ .

Property (H4) and Lemma 4.6 imply that  $H_*$  is of class  $LW_2^p$ . By hypothesis,  $f$  is of class  $LW_2^p$ , so by Lemma 2.4,  $F$  is of class  $LW_2^p$ . The theorem follows.  $\square$

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